Signal Analysis & Communication ECE 355

Ch. 2.3: Properties of LTI Systems

Lecture 4

27-09-2023

Ch. 2.3: PROPERTIES OF LTI SYSTEMS

- Recall that a (CT/DT) LTI system is completely characterized by an impulse response, $h(t) / h[n]$.
  (that's not the case with non-linear & time variant systems)

- LTI systems possess a number of useful properties not possessed by other systems.

① COMMUTATIVE PROPERTY (Convolution is commutative)

CT: $$x(t) * h(t) = h(t) * x(t)$$

Proof

LHS $$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

let $s = (t-\tau)$

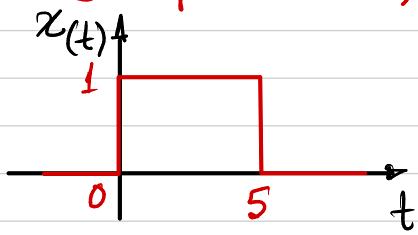
$ds = -d\tau$; limits: $\infty \to -\infty$ | $\Rightarrow ds$ with limits: $-\infty \to \infty$

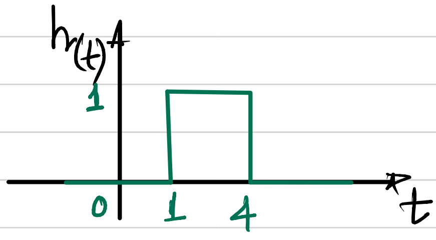
$$\therefore x(t) * h(t) = \int_{-\infty}^{\infty} x(t-s)\, h(s)\, ds$$

$$= \int_{-\infty}^{\infty} h(s)\, x(t-s)\, ds = \text{RHS}$$

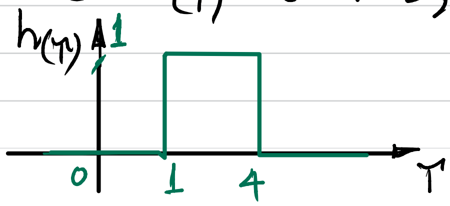
DT: Similar.

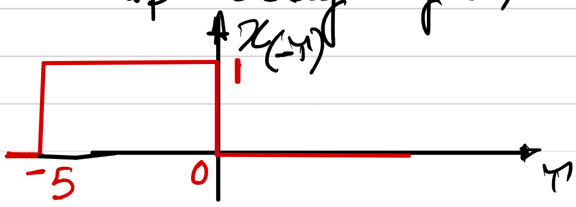
Example ①
(Ex. ② of prev. lecture)



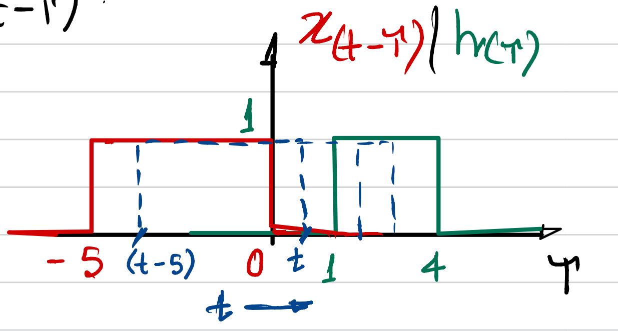


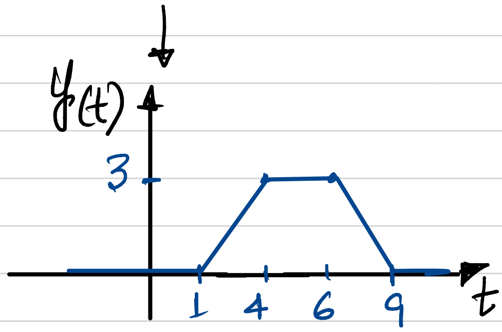
Take $h(\tau)$ as it is, & flip & delay (by 't') $x(\tau)$ here.





Slide $x(-\tau)$ by 't' for different regions of overlap of $h(\tau)$ & $x(t-\tau)$.





Same result as previous.

Example ②

$$h(t) = \delta(t)$$

$$x(t) \longrightarrow \boxed{\delta(t)} \longrightarrow y(t)$$

By commutative property.

$$\delta(t) \longrightarrow \boxed{x(t)} \longrightarrow y(t) = x(t)$$

We know when
$x(t) = \delta(t)$
$y(t) = h(t)$

CALLED IDENTITY SYSTEM

Basically,
$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t)$$

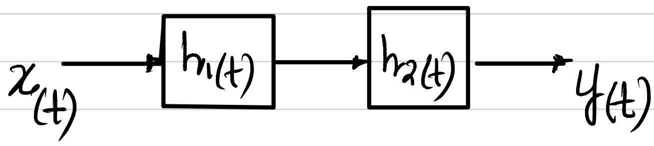
'Sifting' of unit impulse.

$$= \int_{-\infty}^{\infty} \delta(\tau)\,x(t-\tau)\,d\tau = x(t)$$

exists at $\tau = 0$

② ASSOCIATIVE PROPERTY (Convolution is associative)

CT:



$$y(t) = [x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$$

Proof:

$$\text{LHS} = \Big[\underbrace{\int_{-\infty}^{\infty} x(\tau)\, h_1(t-\tau)\, d\tau}_{x_1(t)}\Big] * h_2(t)$$

$$= \int_{-\infty}^{\infty} \Big[\int_{-\infty}^{\infty} x(\tau)\, h_1(s-\tau)\, d\tau\Big]\, h_2(t-s)\, ds$$

$$= \int_{-\infty}^{\infty} x(\tau) \Big[\int_{-\infty}^{\infty} h_1(s-\tau)\, h_2(t-s)\, ds\Big] d\tau \quad \text{— (1)}$$

$$\text{RHS} = \int_{-\infty}^{\infty} x(\tau)\, \underbrace{[h_1 * h_2]_{(t-\tau)}}\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \Big[\int_{-\infty}^{\infty} h_1(\tau')\, h_2(t-\tau-\tau')\, d\tau'\Big] d\tau$$

NOTE: $y(t) = (x * h)(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$$y(t-1) = \boxed{\underbrace{(x*h)(t-1)} \overset{\checkmark}{=} x(t) * h(t-1)} = \int_{-\infty}^{\infty} x(\tau)\cdot h((t-1)-\tau)\, d\tau$$

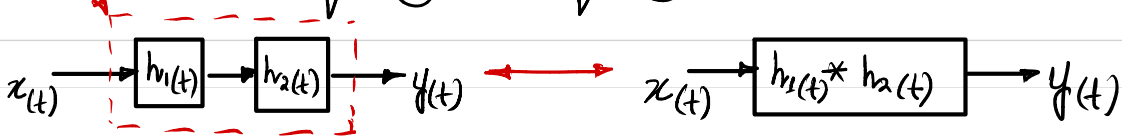
$\neq x(t-1) * h(t-1)$

Now let $\tau' = s - \tau$, $d\tau' = ds$; limits: same

$$= \int_{-\infty}^{\infty} x(\tau) \Big[\int_{-\infty}^{\infty} h_1(s-\tau)\, h_2(t-s)\, ds\Big] d\tau \quad \text{- (2)}$$

System in Series

Eqn (1) = Eqn (2)



NOTE: 1. We can simply write $x(t) * h_1(t) * h_2(t)$

2. Commutative + Associative

$$x(t) * h_1(t) * h_2(t) = x(t) * h_2(t) * h_1(t)$$

3. Extend to any number of systems in series

$$x(t) * h_1(t) * h_2(t) * \dots * h_n(t) = y(t)$$

DT : Similar

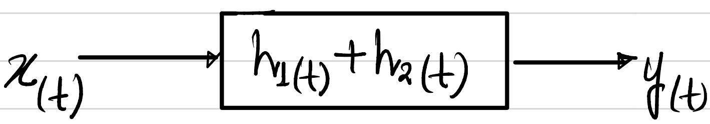
③ DISTRIBUTIVE PROPERTY (Convolution is distributive)

CT: $$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

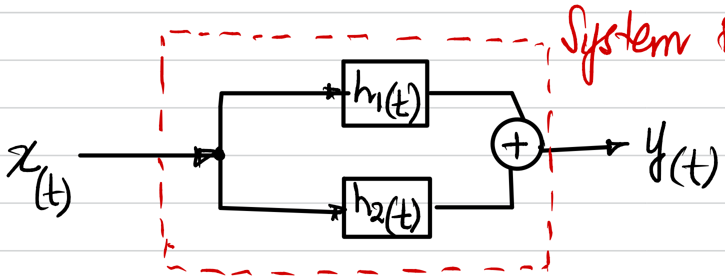
Proof

I. Using LHS = RHS , $\int_{-\infty}^{\infty} x(\tau) \dots d\tau = \dots$

II. View $x(t)$ as sig. & $[h_1(t) + h_2(t)]$ as sys.



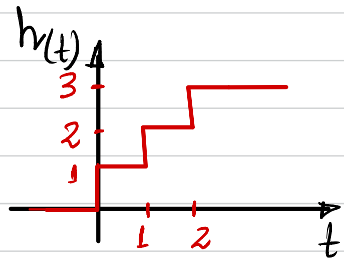
System in parallel



Example

$$h(t) = u(t) + u(t-1) + u(t-2)$$

$$x(t) = e^{-at} u(t)$$



Find $y(t) = x(t) * h(t)$

$$x(t) * u(t) = \frac{1}{a}(1 - e^{-at}) u(t) \quad \text{(Example in lecture 8)}$$

$$x(t) * u(t-1) = \frac{1}{a}(1 - e^{-a(t-1)}) u(t-1)$$

$$x_{(t)} * u_{(t-2)} = \frac{1}{a}\left(1 - e^{-a(t-2)}\right)u_{(t-2)}$$

Therefore,

$$y_{(t)} = x_{(t)} * h_{(t)} = \frac{1}{a}\left[\left(1 - e^{-at}\right)u_{(t)} + \left(1 - e^{-a(t-1)}\right)u_{(t-1)} \cdots\right]$$